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弹性地基上变厚度圆板在集中载荷下的非线性弯曲*

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摘　要

本文用修正迭代法研究了弹性地基上变厚度圆板在集中载荷下的非线性弯曲问题，得到了二次渐近解。

一、引　言

关于弹性地基上圆薄板的大挠度问题，已有很多文章进行了研究[1][2][3]，但研究的都是等厚度板。在工程上，为了充分利用材料的强度，常采用变厚度板，而变厚度板较之于等厚度板，计算更为复杂。作者曾利用修正迭代法对均布载荷作用下 Winkler 地基上变厚度圆板的非线性弯曲问题进行了求解[4]，计算结果表明，对这个问题，修正迭代法是一种行之有效的方法。

本文是文〔4〕工作的继续，仍采用修正迭代法，求解中心集中载荷作用下 Winkler 地基上变厚度圆板的非线性弯曲问题，得到了二次渐近解。最后，把问题退化为 Winkler 地基上的等厚度圆板，与文〔3〕的精确解比较，其精度可以令人满意。

二、基本方程

考虑 Winkler 地基上半径为a的圆板，厚度 $h=h(r)$，本文取厚度沿半径方向为线性变化，即

$$h(r)=h_0\left(1+\varepsilon\frac{r}{a}\right) \tag{2.1}$$

其中 h_0 为圆板中心厚度，ε 为变厚度参数，且 $|\varepsilon|<1$。则 Winkler 地基上的变厚度圆板在中心集中载荷作用下非线性弯曲问题的平衡方程（除中心外）在无量纲化后为

$$\begin{aligned}(1+\varepsilon Kx)^3L_1(u)+u=&K\frac{1}{x}\frac{d}{dx}\left(S_r\frac{du}{dx}\right)-6\varepsilon^2K^2(1+\varepsilon Kx)\left(\frac{d^2u}{dx^2}+\frac{\mu}{x}\frac{du}{dx}\right)-\\&-3\varepsilon K(1+\varepsilon Kx)^2\left(2\frac{d^3u}{dx^3}+\frac{2+\mu}{x}\frac{d^2u}{dx^2}-\frac{1}{x^2}\frac{du}{dx}\right)\end{aligned} \tag{2.2.1}$$

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$$L_2(xS_r)=\frac{\varepsilon K}{1+\varepsilon Kx}\left(x\frac{dS_r}{dx}-\mu S_r\right)-\beta K(1+\varepsilon Kx)\left(\frac{du}{dx}\right)^2 \tag{2.2.2}$$

其中 $$L_1=\frac{1}{x}\frac{d}{dx}x\frac{d}{dx}\frac{1}{x}\frac{d}{dx}x\frac{d}{dx} \tag{2.3.1}$$

$$L_2=x\frac{d}{dx}\frac{1}{x}\frac{d}{dx} \tag{2.3.2}$$

其余各无量纲量的意义见文〔4〕。

为了使所讨论的问题有确定的解，我们取圆板周边为固定夹紧，则无量纲化后的边界条件为

$$x=\frac{1}{K}\text{时，}u=0,\ \frac{du}{dx}=0,\ \frac{dS_r}{dx}-\frac{\mu}{x}S_r=0 \tag{2.4.1}$$

$$x=0\text{ 时，}u\text{ 有限，}S_r=0 \tag{2.4.2}$$

这样，我们的问题就化为在边界条件（2.4）下求解非线性方程组（2.2）。

三、基本方程的求解

我们用修正迭代法来求解边值问题(2.2)和(2.4)。在一次近似中，我们在方程(2.2.1)中略去含 S_r 和 ε 的项，在（2.2.2）中略去方程右端的第一项，得到下列线性边值问题：

$$L_1(u_1)+u_1=0 \tag{3.1.1}$$

$$L_2(xS_{r1})=-\beta K(1+\varepsilon Kx)\left(\frac{du_1}{dx}\right)^2 \tag{3.1.2}$$

$$x=\frac{1}{K}\text{ 时，}u_1=0,\ \frac{du_1}{dx}=0,\ \frac{dS_{r1}}{dx}-\frac{\mu}{x}S_{r1}=0 \tag{3.2.1}$$

$$x=0\text{ 时，}u_1\text{ 有限，}S_{r1}=0 \tag{3.2.2}$$

方程（3.1.1）的解一般可用 Bassel 函数表示，这里采用幂级数解法[5]，即取

$$u_1=c_1\varphi_1(x)+c_2\varphi_2(x)+c_3\varphi_3(x)+c_4\varphi_4(x) \tag{3.3}$$

其中 $$\varphi_1(x)=1-\frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4+\frac{1}{(4!)^2}\left(\frac{x}{2}\right)^8-\frac{1}{(6!)^2}\left(\frac{x}{2}\right)^{12}+\cdots \tag{3.4.1}$$

$$\varphi_2(x)=\left(\frac{x}{2}\right)^2-\frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6+\frac{1}{(5!)^2}\left(\frac{x}{2}\right)^{10}-\frac{1}{(7!)^2}\left(\frac{x}{2}\right)^{14}+\cdots \tag{3.4.2}$$

$$\varphi_3(x)=\varphi_1(x)\ln x+\psi_3(x) \tag{3.4.3}$$

$$\varphi_4(x)=\varphi_2(x)\ln x+\psi_4(x) \tag{3.4.4}$$

$$\psi_3(x)=\frac{S(2)}{(2!)^2}\left(\frac{x}{2}\right)^4-\frac{S(4)}{(4!)^2}\left(\frac{x}{2}\right)^8+\frac{S(6)}{(6!)^2}\left(\frac{x}{2}\right)^{12}-\cdots \tag{3.4.5}$$

$$\psi_4(x)=-\left(\frac{x}{2}\right)^2+\frac{S(3)}{(3!)^2}\left(\frac{x}{2}\right)^6-\frac{S(5)}{(5!)^2}\left(\frac{x}{2}\right)^{10}+\cdots \tag{3.4.6}$$

这里　$S(n)=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$　(3.4.7)

可以直接验证，$\varphi_1(x)\sim\varphi_4(x)$ 线性无关，且各自满足方程(3.1.1)，因而它们的线性组合就是方程(3.1.1)的通解。

由 $x=0$ 时 u_1 应有限的条件，可得 $c_3=0$。此外，在圆板中心附近截取无限小圆柱体侧面上所分布的剪力之和应与中心集中载荷P平衡：

$$\int_0^{2\pi}(Q_r r d\theta)_{r=\delta}+P=0 \tag{3.5}$$

其中　$$Q_r=-D\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dW}{dr}-\frac{dD}{dr}\left(\frac{d^2W}{dr^2}+\frac{\mu}{r}\frac{dW}{dr}\right) \tag{3.6}$$

将无量纲量代入，令 $\delta\to 0$，并令 $\frac{Pa^2}{2\pi D_0 h_0}=Q$，将(3.3)式代入，即得

$$c_4=KQ \tag{3.7}$$

再利用边界条件(3.2.1)式，即可求得 Winkler 地基上圆板在中心集中载荷下的小挠度解

$$u_1=c_1\varphi_1(x)+c_2\varphi_2(x)+KQ\varphi_4(x) \tag{3.8}$$

其中　$$c_1=\frac{\varphi_2\left(\frac{1}{K}\right)\varphi_4'\left(\frac{1}{K}\right)-\varphi_2'\left(\frac{1}{K}\right)\varphi_4\left(\frac{1}{K}\right)}{\varphi_1\left(\frac{1}{K}\right)\varphi_2'\left(\frac{1}{K}\right)-\varphi_1'\left(\frac{1}{K}\right)\varphi_2\left(\frac{1}{K}\right)}\cdot KQ \tag{3.9.1}$$

$$c_2=\frac{\varphi_4\left(\frac{1}{K}\right)\varphi_1'\left(\frac{1}{K}\right)-\varphi_4'\left(\frac{1}{K}\right)\varphi_1\left(\frac{1}{K}\right)}{\varphi_1\left(\frac{1}{K}\right)\varphi_2'\left(\frac{1}{K}\right)-\varphi_1'\left(\frac{1}{K}\right)\varphi_2\left(\frac{1}{K}\right)}\cdot KQ \tag{3.9.2}$$

我们取圆板的无量纲中心挠度 y_0 为迭代参数，在 $x=0$ 时，有

$$u_1=\frac{y_0}{K}=c_1 \tag{3.10}$$

将 c_1 代入，得

$$KQ=\frac{\varphi_1\left(\frac{1}{K}\right)\varphi_2'\left(\frac{1}{K}\right)-\varphi_1'\left(\frac{1}{K}\right)\varphi_2\left(\frac{1}{K}\right)}{\varphi_2\left(\frac{1}{K}\right)\varphi_4'\left(\frac{1}{K}\right)-\varphi_2'\left(\frac{1}{K}\right)\varphi_4\left(\frac{1}{K}\right)}\cdot\frac{y_0}{K} \tag{3.11}$$

将(3.9)及(3.11)式代入(3.8)式，得

$$u_1=\frac{y_0}{K}[\varphi_1(x)+A\varphi_2(x)+B\varphi_4(x)] \tag{3.12}$$

其中　$$A=\frac{\varphi_4\left(\frac{1}{K}\right)\varphi_1'\left(\frac{1}{K}\right)-\varphi_4'\left(\frac{1}{K}\right)\varphi_1\left(\frac{1}{K}\right)}{\varphi_2\left(\frac{1}{K}\right)\varphi_4'\left(\frac{1}{K}\right)-\varphi_2'\left(\frac{1}{K}\right)\varphi_4\left(\frac{1}{K}\right)} \tag{3.13.1}$$

$$B=\frac{\varphi_1\left(\frac{1}{K}\right)\varphi_2'\left(\frac{1}{K}\right)-\varphi_1'\left(\frac{1}{K}\right)\varphi_2\left(\frac{1}{K}\right)}{\varphi_2\left(\frac{1}{K}\right)\varphi_4'\left(\frac{1}{K}\right)-\varphi_2'\left(\frac{1}{K}\right)\varphi_4\left(\frac{1}{K}\right)} \tag{3.13.2}$$

将(3.12)式代入方程(3.1.2)式，积分后得(以下只取到x^{12}项)

$$
\begin{aligned}
S_{r1}=&-\beta\frac{y_0^2}{K}\Big\{\frac{1}{32}(8A^2-20AB+15B^2)\left(\frac{x}{2}\right)^3-\frac{1}{144}(12A-11B)\left(\frac{x}{2}\right)^5+\\
&+\frac{1}{4608}(48-32A^2-51B^2+88AB)\left(\frac{x}{2}\right)^7+\frac{1}{576}\left(\frac{7}{5}A-\frac{483}{200}B\right)\left(\frac{x}{2}\right)^9-\\
&-\frac{1}{14400}\left(\frac{5}{3}-\frac{11}{6}A^2+\frac{267}{40}AB-\frac{129703}{21600}B^2\right)\left(\frac{x}{2}\right)^{11}+\\
&+B\ln x\Big[\frac{1}{8}(4A-5B)\left(\frac{x}{2}\right)^3-\frac{1}{12}\left(\frac{x}{2}\right)^5-\frac{1}{36}\left(\frac{A}{2}-\frac{11}{16}B\right)\left(\frac{x}{2}\right)^7+\\
&+\frac{7}{2880}\left(\frac{x}{2}\right)^9+\frac{1}{14400}\left(\frac{11}{3}A-\frac{267}{40}B\right)\left(\frac{x}{2}\right)^{11}\Big]+\\
&+B^2\ln^2x\Big[\frac{1}{4}\left(\frac{x}{2}\right)^3-\frac{1}{144}\left(\frac{x}{2}\right)^7+\frac{11}{86400}\left(\frac{x}{2}\right)^{11}\Big]\Big\}-\\
&-\varepsilon\beta y_0^2\Big\{\frac{1}{15}\left(4A^2-\frac{124}{15}AB+\frac{1097}{225}B^2\right)\left(\frac{x}{2}\right)^4-\frac{2}{35}\left(2A-\frac{59}{35}B\right)\left(\frac{x}{2}\right)^6+\\
&+\frac{1}{567}\left(9-6A^2+\frac{337}{21}AB-\frac{11755}{1323}B^2\right)\left(\frac{x}{2}\right)^8+\frac{7}{3564}\left(2A-\frac{337}{99}B\right)\left(\frac{x}{2}\right)^{10}-\\
&-\frac{1}{2059200}\left(400-440A^2+\frac{61946}{39}AB-\frac{23677739}{16731}B^2\right)\left(\frac{x}{2}\right)^{12}+\\
&+B\ln x\Big[\frac{4}{15}\left(2A-\frac{31}{15}B\right)\left(\frac{x}{2}\right)^4-\frac{4}{35}\left(\frac{x}{2}\right)^6-\frac{1}{567}\left(12A-\frac{337}{21}B\right)\left(\frac{x}{2}\right)^8+\\
&+\frac{7}{1782}\left(\frac{x}{2}\right)^{10}+\frac{1}{1029600}\left(440A-\frac{30973}{39}B\right)\left(\frac{x}{2}\right)^{12}\Big]+\\
&+B^2\ln^2x\Big[\frac{4}{15}\left(\frac{x}{2}\right)^4-\frac{2}{189}\left(\frac{x}{2}\right)^8+\frac{1}{4680}\left(\frac{x}{2}\right)^{12}\Big]\Big\}-\\
&-\frac{\beta}{K}y_0^2Hx+\frac{L}{x}
\end{aligned}
\tag{3.14}
$$

其中H和L为积分常数。将边界条件(3.2)式代入，可得$L=0$，以及

$$H=H_1+\varepsilon KH_2 \tag{3.15}$$

其中

$$
\begin{aligned}
H_1=&-\frac{1}{1-\mu}\Big\{\frac{1}{64}\Big[(8A^2-20AB+15B^2)(3-\mu)+4B(4A-5B)\Big]\left(\frac{1}{2K}\right)^2-\\
&-\frac{1}{288}\Big[(12A-11B)(5-\mu)+12B\Big]\left(\frac{1}{2K}\right)^4+\\
&+\frac{1}{9216}\Big[(48-32A^2+88AB-51B^2)(7-\mu)-8B(8A-11B)\Big]\left(\frac{1}{2K}\right)^6+\\
&+\frac{1}{5760}\Big[\left(7A-\frac{483}{40}B\right)(9-\mu)+7B\Big]\left(\frac{1}{2K}\right)^8-\\
&-\frac{1}{28800}\Big[\left(\frac{5}{3}-\frac{11}{6}A^2+\frac{267}{40}AB-\frac{129703}{21600}B^2\right)(11-\mu)-B\Big(\frac{11}{3}A-
\end{aligned}
$$

$$-\frac{267}{40}B\Big)\Big]\Big(\frac{1}{2K}\Big)^{10}-\frac{1}{7257600}\Big[\Big(66A-\frac{10651}{84}B\Big)(13-\mu)+66B\Big]\Big(\frac{1}{2K}\Big)^{12}\Big\}-$$

$$-\frac{B}{1-\mu}\ln\frac{1}{K}\Big\{\frac{1}{4}\Big[\Big(A-\frac{5}{4}B\Big)(3-\mu)+B\Big]\Big(\frac{1}{2K}\Big)^{2}-\frac{1}{24}(5-\mu)\Big(\frac{1}{2K}\Big)^{4}-$$

$$-\frac{1}{144}\Big[\Big(A-\frac{11}{8}B\Big)(7-\mu)+B\Big]\Big(\frac{1}{2K}\Big)^{6}+\frac{7}{5760}(9-\mu)\Big(\frac{1}{2K}\Big)^{8}+$$

$$+\frac{1}{14400}\Big[\Big(\frac{11}{6}A-\frac{267}{80}B\Big)(11-\mu)+\frac{11}{6}B\Big]\Big(\frac{1}{2K}\Big)^{10}-$$

$$-\frac{11}{1209600}(13-\mu)\Big(\frac{1}{2K}\Big)^{12}\Big\}-\frac{B^2}{1-\mu}\ln^2\frac{1}{K}\Big\{\frac{1}{8}(3-\mu)\Big(\frac{1}{2K}\Big)^{2}-$$

$$-\frac{1}{288}(7-\mu)\Big(\frac{1}{2K}\Big)^{6}+\frac{11}{172800}(11-\mu)\Big(\frac{1}{2K}\Big)^{10}\Big\} \tag{3.16.1}$$

$$H_2=-\frac{1}{1-\mu}\Big\{\frac{1}{450}\Big[\Big(60A^2-124AB+\frac{1097}{15}B^2\Big)(4-\mu)+4B(30A-31B)\Big]\Big(\frac{1}{2K}\Big)^{3}-$$

$$-\frac{1}{35}\Big[\Big(2A-\frac{59}{35}B\Big)(6-\mu)+2B\Big]\Big(\frac{1}{2K}\Big)^{5}+$$

$$+\frac{1}{1134}\Big[\Big(9-6A^2+\frac{337}{21}AB-\frac{11755}{1323}B^2\Big)(8-\mu)-B\Big(12A-\frac{337}{21}B\Big)\Big]\Big(\frac{1}{2K}\Big)^{7}+$$

$$+\frac{1}{7128}\Big[\Big(14A-\frac{2359}{99}B\Big)(10-\mu)+14B\Big]\Big(\frac{1}{2K}\Big)^{9}-$$

$$-\frac{1}{2059200}\Big[\Big(200-220A^2+\frac{30973}{39}AB-\frac{23677739}{33462}B^2\Big)(12-\mu)-$$

$$-B\Big(440A-\frac{30973}{39}B\Big)\Big]\Big(\frac{1}{2K}\Big)^{11}\Big\}-$$

$$-\frac{B}{1-\mu}\ln\frac{1}{K}\Big\{\frac{2}{15}\Big[\Big(2A-\frac{31}{15}B\Big)(4-\mu)+2B\Big]\Big(\frac{1}{2K}\Big)^{3}-\frac{2}{35}(6-\mu)\Big(\frac{1}{2K}\Big)^{5}-$$

$$-\frac{1}{1134}\Big[\Big(12A-\frac{337}{21}B\Big)(8-\mu)+12B\Big]\Big(\frac{1}{2K}\Big)^{7}+\frac{1}{3564}(10-\mu)\Big(\frac{1}{2K}\Big)^{9}+$$

$$+\frac{1}{2059200}\Big[\Big(440A-\frac{30973}{39}B\Big)(12-\mu)+440B\Big]\Big(\frac{1}{2K}\Big)^{11}\Big\}-$$

$$-\frac{B^2}{1-\mu}\ln^2\frac{1}{K}\Big[\frac{2}{15}(4-\mu)\Big(\frac{1}{2K}\Big)^{3}-\frac{1}{189}(8-\mu)\Big(\frac{1}{2K}\Big)^{7}+$$

$$+\frac{1}{9360}(12-\mu)\Big(\frac{1}{2K}\Big)^{11}\Big] \tag{3.16.2}$$

在二次近似中，我们有如下线性边值问题：

$$L_1(u_2)+u_2=K\frac{1}{x}\frac{d}{dx}\Big(Sr_1\frac{du_1}{dx}\Big)-3\varepsilon KxL_1(u_1)-3\varepsilon^2K^2x^2L_1(u_1)-$$

$$-\varepsilon^3K^3x^3L_1(u_1)-6\varepsilon^2K^2(1+\varepsilon Kx)\Big(\frac{d^2u_1}{dx^2}+\frac{\mu}{x}\frac{du_1}{dx}\Big)-$$

$$-3\varepsilon K(1+\varepsilon Kx)^2\left(2\frac{d^3u_1}{dx^3}+\frac{2+\mu}{x}\frac{d^2u_1}{dx^2}-\frac{1}{x^2}\frac{du_1}{dx}\right) \tag{3.17}$$

$x=\frac{1}{K}$时，$u_2=0$，$\frac{du_2}{dx}=0$ (3.18.1)

$x=0$时，u_2有限 (3.18.2)

注意到$L_1(u_1)=-u_1$，将u_1及S_{r1}代入方程（3.17）式右端，积分后得

$$u_2=D_1\varphi_1(x)+D_2\varphi_2(x)+D_3\varphi_3(x)+D_4\varphi_4(x)+u_2^*(x) \tag{3.19}$$

上式中$u_2^*(x)$是方程（3.17）的特解，其形式为

$$\begin{aligned}u_2^*(x)=&M_1(x)y_0^3+\varepsilon M_2(x)y_0+\varepsilon M_3(x)y_0^3+\varepsilon^2M_4(x)y_0+\\&+\varepsilon^3M_5(x)y_0\end{aligned} \tag{3.20}$$

其中

$$\begin{aligned}M_1(x)=&-\frac{\beta}{K}\Big\{\frac{H_1}{4}\left(A-\frac{3}{2}B\right)\left(\frac{x}{2}\right)^4-\frac{1}{36}\Big(H_1-\frac{1}{4}A^3+\frac{19}{16}A^2B-2AB^2+\\&-\frac{661}{576}B^3\Big)\left(\frac{x}{2}\right)^6-\frac{1}{576}\Big[H_1\left(2A-\frac{25}{6}B\right)+\frac{5}{4}A^2-\frac{29}{8}AB+\\&+\frac{1153}{384}B^2\Big]\left(\frac{x}{2}\right)^8+\frac{1}{14400}\Big(2H_1-\frac{9}{2}A^3+\frac{227}{20}A^2B-\frac{63271}{3200}AB^2+\\&+\frac{1752751}{144000}B^3+\frac{15}{4}A-\frac{139}{32}B\Big)\left(\frac{x}{2}\right)^{10}+\frac{1}{518400}\Big[H_1\left(3A-\frac{147}{20}B\right)+\\&+\frac{89}{4}A^2-\frac{5549}{80}AB+\frac{497173}{9600}B^2-\frac{15}{2}\Big]\left(\frac{x}{2}\right)^{12}+B\ln x\Big[\frac{H_1}{4}\left(\frac{x}{2}\right)^4+\\&+\frac{1}{36}\left(\frac{3}{4}A^2-\frac{19}{8}AB+2B^2\right)\left(\frac{x}{2}\right)^6-\frac{1}{576}\left(2H_1+\frac{5}{2}A-\frac{29}{8}B\right)\left(\frac{x}{2}\right)^8-\\&-\frac{1}{14400}\left(\frac{27}{4}A^2-\frac{227}{10}AB+\frac{63271}{3200}B^2-\frac{15}{4}\right)\left(\frac{x}{2}\right)^{10}+\\&-\frac{1}{518400}\left(3H_1+\frac{89}{2}A-\frac{5549}{80}B\right)\left(\frac{x}{2}\right)^{12}\Big]+B^2\ln^2x\Big[\frac{1}{36}\Big(\frac{3}{4}A-\\&-\frac{19}{16}B\Big)\left(\frac{x}{2}\right)^6-\frac{5}{2304}\left(\frac{x}{2}\right)^8-\frac{1}{14400}\left(\frac{27}{4}A-\frac{227}{20}B\right)\left(\frac{x}{2}\right)^{10}+\\&+\frac{89}{2073600}\left(\frac{x}{2}\right)^{12}\Big]+B^3\ln^3x\Big[\frac{1}{144}\left(\frac{x}{2}\right)^6-\frac{1}{6400}\left(\frac{x}{2}\right)^{10}\Big]\Big\}\end{aligned} \tag{3.21.1}$$

$$\begin{aligned}M_2(x)=&-\frac{2}{3}\Big[\left(2A+\frac{5}{3}B\right)+\left(2A-\frac{13}{3}B\right)\mu\Big]\left(\frac{x}{2}\right)^3+\frac{2}{25}(11+\mu)\left(\frac{x}{2}\right)^5+\\&+\frac{1}{11025}\Big[\left(1497A-\frac{79993}{35}B\right)+\left(237A-\frac{22558}{35}B\right)\mu\Big]\left(\frac{x}{2}\right)^7-\\&-\frac{1}{1190700}(13849+559\mu)\left(\frac{x}{2}\right)^9-\frac{1}{576298800}\Big[\Big(323989A-\\&-\frac{9573149801}{13860}B\Big)+\left(19869A-\frac{86473831}{1260}B\right)\mu\Big]\left(\frac{x}{2}\right)^{11}+\end{aligned}$$

$$+B\ln X\left[-\frac{4}{3}(1+\mu)\left(\frac{x}{2}\right)^3+\frac{1}{3675}(499+79\mu)\left(\frac{x}{2}\right)^7-\right.$$

$$\left.-\frac{1}{576298800}(323989+22429\mu)\left(\frac{x}{2}\right)^{11}\right] \tag{3.21.2}$$

$$M_3(x)=-\beta\left\{\frac{H_2}{4}\left(A-\frac{3}{2}B\right)\left(\frac{x}{2}\right)^4-\frac{H_2}{36}\left(\frac{x}{2}\right)^6+\frac{16}{3675}\left(A^3-\frac{169}{42}A^2B+\right.\right.$$

$$\left.+\frac{251183}{44100}AB^2-\frac{8570577}{3087000}B^3\right)\left(\frac{x}{2}\right)^7-\frac{H_2}{576}\left(2A-\frac{18145}{6048}B\right)\left(\frac{x}{2}\right)^8-$$

$$-\frac{8}{59535}\left[13A^2-\frac{2017}{63}AB+\frac{952681}{44100}B^2\right)\left(\frac{x}{2}\right)^9+\frac{H_2}{7200}\left(\frac{x}{2}\right)^{10}-$$

$$-\frac{4}{343035}\left(\frac{383}{35}A^3-\frac{3493531}{72765}A^2B+\frac{5909797217}{84043575}AB^2-\right.$$

$$\left.-\frac{2022995337}{56162119}B^3-23A+\frac{17667}{770}B\right)\left(\frac{x}{2}\right)^{11}+$$

$$+\frac{H_2}{518400}\left(3A-\frac{1744853}{362880}B\right)\left(\frac{x}{2}\right)^{12}+B\ln X\left[\frac{H_2}{4}\left(\frac{x}{2}\right)^4+\right.$$

$$+\frac{16}{1225}\left(A^2-\frac{169}{63}AB+\frac{251183}{132300}B^2\right)\left(\frac{x}{2}\right)^7-\frac{H_2}{288}\left(\frac{x}{2}\right)^8-$$

$$-\frac{16}{3969}\left(\frac{13}{15}A-\frac{2017}{1890}B\right)\left(\frac{x}{2}\right)^9+\frac{4}{343035}\left(18-\frac{3097}{105}A^2+\right.$$

$$\left.\left.+\frac{6480437}{72765}AB-\frac{1142885794}{16808715}B^2\right)\left(\frac{x}{2}\right)^{11}+\frac{H_2}{172800}\left(\frac{x}{2}\right)^{12}\right]+$$

$$+B^2\ln^2x\left[\frac{16}{1225}\left(A-\frac{169}{126}B\right)\left(\frac{x}{2}\right)^7-\frac{104}{59535}\left(\frac{x}{2}\right)^9-\right.$$

$$\left.-\frac{4}{12006225}\left(1149A-\frac{3564581}{2079}B\right)\left(\frac{x}{2}\right)^{11}\right]+$$

$$\left.+B^3\ln^3x\left[\frac{16}{3675}\left(\frac{x}{2}\right)^7-\frac{1532}{12006225}\left(\frac{x}{2}\right)^{11}\right]\right\} \tag{3.21.3}$$

$$M_4(x)=3K\left\{-\frac{1}{4}\left[(2A+B)+(2A-3B)\mu\right]\left(\frac{x}{2}\right)^4+\frac{1}{18}(7+\mu)\left(\frac{x}{2}\right)^6+\right.$$

$$+\frac{1}{576}\left[(36A-59B)+\left(4A-\frac{25}{3}B\right)\mu\right]\left(\frac{x}{2}\right)^8-\frac{1}{3600}(19+\mu)\left(\frac{x}{2}\right)^{10}-$$

$$-\frac{1}{518400}\left[\left(6A+\frac{1219}{30}B\right)+\left(6A-\frac{103}{10}B\right)\mu\right]\left(\frac{x}{2}\right)^{12}+$$

$$+B\ln x\left[-\frac{1}{2}(1+\mu)\left(\frac{x}{2}\right)^4+\frac{1}{144}(9+\mu)\left(\frac{x}{2}\right)^8-\right.$$

$$\left.\left.-\frac{1}{259200}(67+3\mu)\left(\frac{x}{2}\right)^{12}\right]\right\} \tag{3.21.4}$$

$$M_5(x)=K^2\Big\{-\frac{8}{375}\Big[(30A+13B)+(30A-37B)\mu\Big]\Big(\frac{x}{2}\Big)^5+$$

$$+\frac{8}{245}(17+3\mu)\Big(\frac{x}{2}\Big)^7+\frac{1}{6251175}\Big[\Big(59428A-\frac{4922641}{5}B\Big)+$$

$$+\Big(60228A-\frac{540291}{5}B\Big)\mu\Big]\Big(\frac{x}{2}\Big)^9-\frac{1}{2401245}(19081+1119\mu)\Big(\frac{x}{2}\Big)^{11}+$$

$$+B\ln x\Big[-\frac{16}{25}(1+\mu)\Big(\frac{x}{2}\Big)^5+\frac{4}{99225}(2339+239\mu)\Big(\frac{x}{2}\Big)^9\Big]\Big\} \quad (3.21.5)$$

由 $x=0$ 时 u_2 应有限的条件，可得（3.19）式中 $D_3=0$。在圆板中心附近截取无限小圆柱体侧面上所分布的剪力之和应与中心集中载荷 P 平衡，通过与（3.6）式类似的处理，可得

$$D_4=KQ \quad (3.22)$$

利用边界条件(3.18)，即可确定另两个积分常数 D_1 及 D_2，于是得到二次渐近解

$$u_2=D_1\varphi_1(x)+D_2\varphi_2(x)+KQ\varphi_4(x)+u_2^*(x) \quad (3.23)$$

利用（3.9）式，可以把 D_1 及 D_2 写成

$$D_1=c_1+\frac{\varphi_2\Big(\frac{1}{K}\Big)u_2^{*\prime}\Big(\frac{1}{K}\Big)-\varphi_2'\Big(\frac{1}{K}\Big)u_2^*\Big(\frac{1}{K}\Big)}{\varphi_1\Big(\frac{1}{K}\Big)\varphi_2'\Big(\frac{1}{K}\Big)-\varphi_1'\Big(\frac{1}{K}\Big)\varphi_2\Big(\frac{1}{K}\Big)} \quad (3.24.1)$$

$$D_2=c_2+\frac{\varphi_1'\Big(\frac{1}{K}\Big)u_2^*\Big(\frac{1}{K}\Big)-\varphi_1\Big(\frac{1}{K}\Big)u_2^{*\prime}\Big(\frac{1}{K}\Big)}{\varphi_1\Big(\frac{1}{K}\Big)\varphi_2'\Big(\frac{1}{K}\Big)-\varphi_1'\Big(\frac{1}{K}\Big)\varphi_2\Big(\frac{1}{K}\Big)} \quad (3.24.2)$$

在 $x=0$ 时，有 $\quad u_2=\frac{y_0}{K}=D_1 \quad (3.25)$

将 (3.24.1) 式代入上式，并注意到 $c_1=\frac{KQ}{B}$，整理后，便得到中心挠度 y_0 与载荷 Q 及变厚度参数 ε 之间的关系式

$$Q=\frac{B}{K^2}y_0+N_1(K)y_0^3+\varepsilon N_2(K)y_0+\varepsilon N_3(K)y_0^3+\varepsilon^2N_4(K)y_0+$$

$$+\varepsilon^3N_5(K)y_0 \quad (3.26)$$

其中：

$$N_1(K)=-\frac{\beta B}{K^2}\Big\{\frac{H_1}{4}\Big[\Big(A-\frac{3}{2}B\Big)(R_2-4R_1)-BR_1\Big]\Big(\frac{1}{2K}\Big)^4-$$

$$-\frac{1}{36}\Big[\Big(H_1-\frac{1}{4}A^3+\frac{19}{16}A^2B-2AB^2+\frac{661}{576}B^3\Big)(R_2-6R_1)+$$

$$+BR_1\Big(\frac{3}{4}A^2-\frac{19}{8}AB+2B^2\Big)\Big]\Big(\frac{1}{2K}\Big)^6-\frac{1}{576}\Big[\Big(2AH_1-\frac{25}{6}BH_1+$$

$$+\frac{5}{4}A^2-\frac{29}{8}AB+\frac{1153}{384}B^2\Big)(R_2-8R_1)-BR_1\Big(2H_1+\frac{5}{2}A-$$

$$-\frac{29}{8}B\Big)\Big]\Big(\frac{1}{2K}\Big)^8+\frac{1}{14400}\Big[\Big(2H_1-\frac{9}{2}A^3+\frac{227}{20}A^2B-\frac{63271}{3200}AB^2+$$

$$+\frac{1752751}{144000}B^3+\frac{15}{4}A-\frac{139}{32}B\Big)(R_2-10R_1)+BR_1\Big(\frac{27}{4}A^2-\frac{227}{10}AB+$$

$$+\frac{63271}{3200}B^2-\frac{15}{4}\Big)\Big]\Big(\frac{1}{2K}\Big)^{10}+\frac{1}{518400}\Big[\Big(3AH_1-\frac{147}{20}BH_1+$$

$$+\frac{89}{4}A^2-\frac{5549}{80}AB+\frac{497173}{9600}B^2-\frac{15}{2}\Big)(R_2-12R_1)-BR_1\Big(3H_1+$$

$$+\frac{89}{2}A-\frac{5549}{80}B\Big)\Big]\Big(\frac{1}{2K}\Big)^{12}\Big\}-\frac{\beta B^2}{K^2}\ln\frac{1}{K}\Big\{\frac{H_1}{4}(R_2-4R_1)\Big(\frac{1}{2K}\Big)^4+$$

$$+\frac{1}{36}\Big[\Big(\frac{3}{4}A^2-\frac{19}{8}AB+2B^2\Big)(R_2-6R_1)-2BR_1\Big(\frac{3}{4}A-$$

$$-\frac{19}{16}B\Big)\Big]\Big(\frac{1}{2K}\Big)^6-\frac{1}{576}\Big[\Big(2H_1+\frac{5}{2}A-\frac{29}{8}B\Big)(R_2-8R_1)-$$

$$-\frac{5}{2}BR_1\Big]\Big(\frac{1}{2K}\Big)^8-\frac{1}{14400}\Big[\Big(\frac{27}{4}A^2-\frac{227}{10}AB+\frac{63271}{3200}B^2-$$

$$-\frac{15}{4}\Big)(R_2-10R_1)-BR_1\Big(\frac{27}{2}A-\frac{227}{10}B\Big)\Big]\Big(\frac{1}{2K}\Big)^{10}+$$

$$+\frac{1}{518400}\Big[\Big(3H_1+\frac{89}{2}A-\frac{5549}{80}B\Big)(R_2-12R_1)-\frac{89}{2}BR_1\Big]\Big(\frac{1}{2K}\Big)^{12}\Big\}-$$

$$-\frac{\beta B^3}{K^2}\ln^2\frac{1}{K}\Big\{\frac{1}{36}\Big[\Big(\frac{3}{4}A-\frac{19}{16}B\Big)(R_2-6R_1)-\frac{3}{4}BR_1\Big]\Big(\frac{1}{2K}\Big)^6-$$

$$-\frac{5}{2304}(R_2-8R_1)\Big(\frac{1}{2K}\Big)^8-\frac{1}{14400}\Big[\Big(\frac{27}{4}A-\frac{227}{20}B\Big)(R_2-10R_1)-$$

$$-\frac{27}{4}BR_1\Big]\Big(\frac{1}{2K}\Big)^{10}+\frac{89}{2073600}(R_2-12R_1)\Big(\frac{1}{2K}\Big)^{12}\Big\}-$$

$$-\frac{\beta B^4}{K^2}\ln^3\frac{1}{K}\Big[\frac{1}{144}(R_2-6R_1)\Big(\frac{1}{2K}\Big)^6-$$

$$-\frac{1}{6400}(R_2-10R_1)\Big(\frac{1}{2K}\Big)^{10}\Big] \tag{3.27.1}$$

$$N_2(K)=-\frac{4}{9}B\Big\{[(6A+5B)+(6A-13B)\mu](R_2-3R_1)-6BR_1(1+\mu)\Big\}\Big(\frac{1}{2K}\Big)^4+$$

$$+\frac{4}{25}(11+\mu)B(R_2-5R_1)\Big(\frac{1}{2K}\Big)^6+\frac{2B}{11025}\Big\{\Big[\Big(1497A-\frac{79993}{35}B\Big)+$$

$$+\Big(237A-\frac{22558}{35}B\Big)\mu\Big](R_2-7R_1)-3(499+79\mu)BR_1\Big\}\Big(\frac{1}{2K}\Big)^8-$$

$$-\frac{B}{595350}(13849+559\mu)(R_2-9R_1)\Big(\frac{1}{2K}\Big)^{10}-$$

$$-\frac{B}{288149400}\Big\{\Big[\Big(323989A-\frac{9573149801}{13860}B\Big)+\Big(19869A-$$

$$-\frac{86473831}{1260}B\Big)\mu\Big](R_2-11R_1)-BR_1(323989+22429\mu)\Big\}\Big(\frac{1}{2K}\Big)^{12}+$$

$$+B^2\ln\frac{1}{K}\Big[-\frac{8}{3}(1+\mu)(R_2-3R_1)\Big(\frac{1}{2K}\Big)^4+$$

$$+\frac{2}{3675}(499+79\mu)(R_2-7R_1)\Big(\frac{1}{2K}\Big)^8+$$

$$-\frac{1}{288149400}(323989+22429\mu)(R_2-11R_1)\Big(\frac{1}{2K}\Big)^{12}\Big] \quad (3.27.2)$$

$$N_3(K)=-\frac{\beta B}{K}\Big\{\frac{H_2}{4}\Big[\Big(A-\frac{3}{2}B\Big)(R_2-4R_1)-BR_1\Big]\Big(\frac{1}{2K}\Big)^4-$$

$$-\frac{H_2}{36}(R_2-6R_1)\Big(\frac{1}{2K}\Big)^6+\frac{16}{3675}\Big[\Big(A^3-\frac{169}{42}A^2B+\frac{251183}{44100}AB^2-$$

$$-\frac{8570577}{3087000}B^3\Big)(R_2-7R_1)-3BR_1\Big(A^2-\frac{169}{63}AB+\frac{251183}{132300}B^2\Big)\Big]\Big(\frac{1}{2K}\Big)^7-$$

$$-\frac{H_2}{576}\Big[\Big(2A-\frac{18145}{6048}B\Big)(R_2-8R_1)-2BR_1\Big]\Big(\frac{1}{2K}\Big)^8-$$

$$-\frac{8}{59535}\Big[\Big(13A^2-\frac{2017}{63}AB+\frac{952681}{44100}B^2\Big)(R_2-9R_1)+$$

$$+2BR_1\Big(13A-\frac{2017}{126}B\Big)\Big]\Big(\frac{1}{2K}\Big)^9+\frac{H_2}{7200}(R_2-10R_1)\Big(\frac{1}{2K}\Big)^{10}-$$

$$-\frac{4}{343035}\Big[\Big(\frac{383}{35}A^3-\frac{3493531}{72765}A^2B+\frac{5909797217}{84043575}AB^2-$$

$$-\frac{2022995337}{56162119}B^3-23A+\frac{17667}{770}B\Big)(R_2-11R_1)+$$

$$+BR_1\Big(18-\frac{3097}{105}A^2+\frac{6480437}{72765}AB-\frac{1142885794}{16808715}B^2\Big)\Big]\Big(\frac{1}{2K}\Big)^{11}+$$

$$+\frac{H_2}{518400}\Big[\Big(3A-\frac{1744853}{362880}B\Big)(R_2-12R_1)-3BR_1\Big]\Big(\frac{1}{2K}\Big)^{12}\Big\}-$$

$$-\frac{\beta B^2}{K}\ln\frac{1}{K}\Big\{\frac{H_2}{4}(R_2-4R_1)\Big(\frac{1}{2K}\Big)^4+\frac{16}{1225}\Big[\Big(A^2-\frac{169}{63}AB+$$

$$+\frac{251183}{132300}B^2\Big)(R_2-7R_1)--2BR_1\Big(A-\frac{169}{126}B\Big)\Big]\Big(\frac{1}{2K}\Big)^7-$$

$$-\frac{H_2}{288}(R_2-8R_1)\Big(\frac{1}{2K}\Big)^8-\frac{8}{59535}\Big[\Big(26A-\frac{2017}{63}B\Big)(R_2-9R_1)-$$

$$-26BR_1\Big]\Big(\frac{1}{2K}\Big)^9+\frac{4}{343035}\Big[\Big(18-\frac{3097}{105}A^2+\frac{6480437}{72765}AB-$$

$$-\frac{1142885794}{16808715}B^2\Big)(R_2-11R_1)+\frac{2BR_1}{35}\Big(1149A-$$

$$-\frac{3564581}{2079}B\Big)\Big]\left(\frac{1}{2K}\right)^{11}+\frac{BH_2}{172800}(R_2-12R_1)\left(\frac{1}{2K}\right)^{12}\Big\}-$$

$$-\frac{\beta B^3}{K}\ln^2\frac{1}{K}\Big\{\frac{16}{3675}\Big[\Big(3A-\frac{169}{42}B\Big)(R_2-7R_1)-\frac{3}{2}BR_1\Big]\left(\frac{1}{2K}\right)^7-$$

$$-\frac{104}{59535}(R_2-9R_1)\left(\frac{1}{2K}\right)^9-\frac{4}{12006225}\Big[\Big(1149A-$$

$$-\frac{3564581}{2079}B\Big)(R_2-11R_1)-\frac{1149}{2}BR_1\Big]\left(\frac{1}{2K}\right)^{11}\Big\}-$$

$$-\frac{\beta B^4}{K}\ln^3\frac{1}{K}\Big[\frac{16}{3675}(R_2-7R_1)\left(\frac{1}{2K}\right)^7-$$

$$-\frac{1532}{12006225}(R_2-11R_1)\left(\frac{1}{2K}\right)^{11}\Big] \tag{3.27.3}$$

$$N_4(K)=-\frac{3}{4}B\Big\{[(2A+B)+(2A-3B)\mu](R_2-4R_1)-2BR_1(1+\mu)\Big\}\left(\frac{1}{2K}\right)^4+$$

$$+\frac{B}{6}(7+\mu)(R_2-6R_1)\left(\frac{1}{2K}\right)^6+\frac{B}{192}\Big\{\Big[(36A-59B)+$$

$$+\Big(4A-\frac{25}{3}B\Big)\mu\Big](R_2-8R_1)-4BR_1(9+\mu)\Big\}\left(\frac{1}{2K}\right)^8-$$

$$-\frac{B}{1200}(19+\mu)(R_2-10R_1)\left(\frac{1}{2K}\right)^{10}-\frac{B}{172800}\Big\{\Big[\Big(6A+\frac{1219}{30}B\Big)+$$

$$+\Big(6A-\frac{103}{10}B\Big)\mu\Big](R_2-12R_1)-2BR_1(67+3\mu)\Big\}\left(\frac{1}{2K}\right)^{12}+$$

$$+B^2\ln\frac{1}{K}\Big[-\frac{3}{2}(1+\mu)(R_2-4R_1)\left(\frac{1}{2K}\right)^4+\frac{1}{48}(9+\mu)(R_2-$$

$$-8R_1)\left(\frac{1}{2K}\right)^8-\frac{1}{86400}(67+3\mu)(R_2-12R_1)\left(\frac{1}{2K}\right)^{12}\Big] \tag{3.27.4}$$

$$N_5(K)=-\frac{8BK}{375}\Big\{[(30A+13B)+(30A-37B)\mu](R_2-5R_1)-$$

$$-30BR_1(1+\mu)\Big\}\left(\frac{1}{2K}\right)^5+\frac{8BK}{245}(17+3\mu)(R_2-7R_1)\left(\frac{1}{2K}\right)^7+$$

$$+\frac{4BK}{99225}\Big\{\Big[\Big(\frac{14857}{63}A-\frac{4922641}{1260}B\Big)+\Big(239A-$$

$$-\frac{180097}{420}B\Big)\mu\Big](R_2-9R_1)-BR_1(2339+239\mu)\Big\}\left(\frac{1}{2K}\right)^9-$$

$$-\frac{BK}{2401245}(19081+1119\mu)(R_2-11R_1)\left(\frac{1}{2K}\right)^{11}+$$

$$+B^2K\ln\frac{1}{K}\Big[-\frac{16}{25}(1+\mu)(R_2-5R_1)\left(\frac{1}{2K}\right)^5+$$

$$+\frac{4}{99225}(2339+239\mu)(R_2-9R_1)\left(\frac{1}{2K}\right)^9\Big] \tag{3.27.5}$$

式中：

$$R_1=\frac{K\varphi_2\left(\frac{1}{K}\right)}{\varphi_1\left(\frac{1}{K}\right)\varphi_2'\left(\frac{1}{K}\right)-\varphi_1'\left(\frac{1}{K}\right)\varphi_2\left(\frac{1}{K}\right)} \tag{3.28.1}$$

$$R_2=\frac{\varphi_2'\left(\frac{1}{K}\right)}{\varphi_1\left(\frac{1}{K}\right)\varphi_2'\left(\frac{1}{K}\right)-\varphi_1'\left(\frac{1}{K}\right)\varphi_2\left(\frac{1}{K}\right)} \tag{3.28.2}$$

我们以 $K=0.8$，$\mu=0.3$ 的情况为例，这时 (3.26) 式成为

$$Q=8.4838y_0+3.7692y_0^3+12.3689\varepsilon y_0+1.6396\varepsilon y_0^3+ \\ +8.8396\varepsilon^2y_0+2.4396\varepsilon^3y_0 \tag{3.29}$$

对于不同的 ε 值，即可根据上式画出 Q 与 y_0 的特征曲线（图 1 ）。由图 1 可见，对于相同的 Q 值，ε 越大，圆板的挠度就越小，这是合理的。

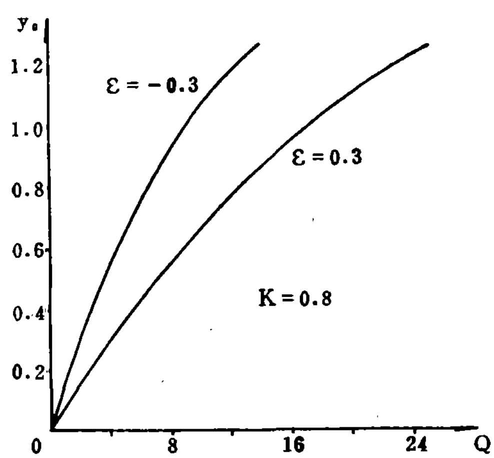

图 1

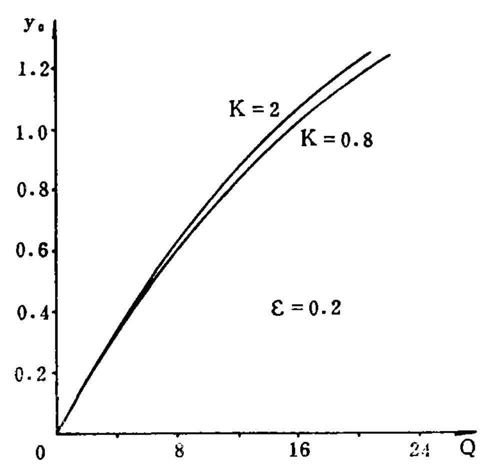

图 2

在 (3.26) 式中取 $\varepsilon=0.1$，则 (3.26) 式成为

$$Q=\left(\frac{B}{K^2}+0.1N_2+0.01N_4+0.001N_6\right)y_0+(N_1+0.1N_3)y_0^3 \tag{3.30}$$

仍取 $\mu=0.3$，则可根据上式画出取不同 K 值时 Q 与 y_0 的特征曲线（图2 ）。由图 2 可见，对于相同的 Q 值，随着 K 的增大，中心挠度 y_0 亦随之增大。K 值越大，意味着地基越柔软，因此 K 值对挠度的影响也是合理的。

为了与文〔3〕的精确解比较，我们令 $\varepsilon=0$，这时问题退化为 Winkler 地基上等厚度圆板在中心集中载荷作用下的非线性弯曲问题，在 (3.26) 式中取 $\mu=0.3$，$Q=19.3673$，$K=0.5$（相当于文〔3〕中 $p=4.0$，$\overline{k}=1$ 的情形），由 (3.26) 式可求得 $y_0=1.2602$（计算到 K^{18} 项），换算成文〔3〕中的量，为 $w_0=2.0822$，这个结果与精确解比较，误差仅为 4 %，可以令人满意。

四、结　语

1．本文采用修正迭代法计算 Winkler 地基上变厚度圆板的非线性弯曲问题，其二次渐近解便有令人满意的精确度，因此这种方法在一定范围内是有效的，可以应用于工程上。

2．由计算过程可知，求解高次修正迭代解的计算工作将极其繁复，以至于几乎是不可能的。近年来出现了解析电算法，利用电子计算机进行符号运算，可以求得高次修正迭代解。不仅把人们从复杂的计算中解放出来，而且可以达到人们所希望的精度。

3．本文渐近解中级数的收敛性问题，因级数系数的规律不易找到，难以求得其收敛范围。但收敛性问题直接关系到这个方法的适用范围，因此需要对这个问题作进一步的研究。

4．本文仅考虑了厚度沿半径方向为线性变化的变厚度圆板，对于厚度沿半径方向按其它规律变化的变厚度圆板，只要其厚度变化较为平缓，也可仿此求得渐近解。

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Nonlinear Bending of Circular Plates With Variable Thickness On Elastic Foundations Under Central Concentrated Load

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ABSTRACT

In this paper, the nonlinearly bending problems of circular plates with variable thickness on elastic foundations under central concentrated load is investigated by use of the modified iteration method, and a second order asymptotic sulution is derived.